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Symmetries of the Fokker–Planck equation with a constant diffusion matrix in 2 + 1 dimensions

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Abstract. We completely classify the symmetries of the Fokker–Planck equation in two spatial dimensions with a constant positive-definite diffusion matrix. We apply these results to construct group-invariant solutions for a physically interesting family of Fokker–Planck equations.

1. Preliminaries

The time evolution of a stochastic process with continuous motion is described by the Fokker–Planck equation:

$$\partial_t p(\boldsymbol{x}, t) = -\sum_{i=1}^{n} \partial_i [A_i(\boldsymbol{x}) p(\boldsymbol{x}, t)] + \tfrac{1}{2} \sum_{i,j=1}^{n} \partial_i \partial_j [B_{ij}(\boldsymbol{x}) p(\boldsymbol{x}, t)] \tag{1}$$

where the vector $\boldsymbol{A} = (A_1, \ldots, A_n)$ and the symmetric matrix $\mathbf{B} = (B_{ij})$ are known as the *drift vector* and the *diffusion matrix*, respectively. The importance of the Fokker–Planck equation, which arises in a large variety of phenomena in physics, chemistry, and biology, [1–3], cannot be overestimated. The determination of the local Lie symmetries of the Fokker–Planck equation—which, in turn, may be used to find group-invariant solutions by solving a differential equation in fewer independent variables [4,5]—is therefore a relevant problem. In one spatial dimension complete results are known. The symmetries of some particular cases of the Fokker–Planck equation were first analysed in [6–8]. Cicogna and Vitali [9, 10] and Rudra [11], provided an exhaustive classification of the symmetries of a generic Fokker–Planck equation in one spatial dimension. The situation in higher dimensions is remarkably different. In fact, to the best of the author's knowledge, the only case which has been studied in the literature is a special form of Kramers' equation in 2 + 1 dimensions, for which the diffusion matrix is constant and degenerate [8]. The aim of this paper is precisely to study the symmetries of the Fokker–Planck equation in two spatial dimensions with a constant positive-definite diffusion matrix. It should be emphasized that the diffusion matrix is constant in many situations of physical interest, e.g. in a Rayleigh or in an Ornstein–Uhlenbeck process. Any Fokker–Planck equation (1) possesses the *trivial* symmetries associated to the time-translational invariance and linearity of the equation. We shall obtain in what follows explicit conditions for the existence of nontrivial symmetries of the Fokker–Planck equation. Then, in section 2, we determine the general solution of these conditions in the particular case in which the drift vector $\boldsymbol{A}$ is irrotational and the Fokker–Planck equation is equivalent to the Schrödinger equation (with

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imaginary time). We completely classify the Fokker–Planck equations admitting nontrivial symmetries, and we present the corresponding symmetry generators. Our results are consistent with the classification of the symmetries of the time-dependent Schrödinger equation with a time-independent potential previously found by Boyer [12]. In section 3 we present the classification of the symmetries in the case in which the drift vector is *not* irrotational. Finally, in section 4 we make use of our classification to construct explicit group-invariant solutions for a physically relevant class of Fokker–Planck equations.

Under the above assumptions, we can rewrite the Fokker–Planck equation (1) in the form†

$$u_t = -\nabla \cdot (\boldsymbol{A}u) + \frac{\beta}{2}\Delta u \tag{2}$$

where β is a nonzero real constant. We shall take $\beta = 1$ without any loss of generality. Following the well known Olver procedure [4], our purpose is to determine all functions ξ, η, τ and ϕ such that the second prolongation of the vector field

$$X = \xi(x, y, t, u)\partial_x + \eta(x, y, t, u)\partial_y + \tau(x, y, t, u)\partial_t + \phi(x, y, t, u)\partial_u \tag{3}$$

is tangent to the solution manifold of equation (2). Equating to zero the coefficients of the monomials in u and its derivatives containing second- and third-order derivatives, one immediately obtains

$$\tau = \tau(t) \qquad \xi = \frac{\tau_t}{2}x + \xi_1(t)y + \xi_0(t) \qquad \eta = \frac{\tau_t}{2}y - \xi_1(t)x + \eta_0(t). \tag{4}$$

The coefficients of the terms in u_x^2 and u_y^2 now imply that

$$\phi = \phi_1(x, y, t)u + \phi_0(x, y, t) \tag{5}$$

where ϕ_0 must be a solution of the Fokker–Planck equation (2). The remaining three conditions for u_x, u_y and u then lead to the equations

$$\nabla\phi_1 = -\frac{\tau_{tt}}{2}\boldsymbol{r} + \frac{\tau_t}{2}(1 + \boldsymbol{r}\cdot\nabla)\boldsymbol{A} - \xi_{1t}J\boldsymbol{r} - \xi_1(J + \boldsymbol{r}\wedge\nabla)\boldsymbol{A} - \boldsymbol{\rho}_t + (\boldsymbol{\rho}\cdot\nabla)\boldsymbol{A} \tag{6}$$

$$\phi_{1t} = \frac{\tau_{tt}}{2}(\boldsymbol{r}\cdot\boldsymbol{A} - 1) - \frac{\tau_t}{2}(2M + \boldsymbol{r}\cdot\nabla M) - \xi_{1t}\boldsymbol{r}\wedge\boldsymbol{A} + \xi_1\boldsymbol{r}\wedge\nabla M + \boldsymbol{\rho}_t\cdot\boldsymbol{A} - \boldsymbol{\rho}\cdot\nabla M \tag{7}$$

where

$$\boldsymbol{r} = (x, y) \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \boldsymbol{\rho} = (\xi_0, \eta_0) \qquad M = \tfrac{1}{2}(\boldsymbol{A}^2 + \nabla\cdot\boldsymbol{A}).$$

Note that in two dimensions $\boldsymbol{v}\wedge\boldsymbol{w} = v_1w_2 - v_2w_1$ is a scalar. Imposing that $\phi_{1xy} = \phi_{1yx}$, $\phi_{1xt} = \phi_{1tx}$ and $\phi_{1yt} = \phi_{1ty}$, we obtain the compatibility conditions

$$-2\xi_{1t} = \frac{\tau_t}{2}(2B + \boldsymbol{r}\cdot\nabla B) - \xi_1\boldsymbol{r}\wedge\nabla B + \boldsymbol{\rho}\cdot\nabla B \tag{8}$$

$$\begin{aligned} \frac{\tau_{ttt}}{2}\boldsymbol{r} - \frac{\tau_t}{2}(3 + \boldsymbol{r}\cdot\nabla)\nabla M + \xi_{1tt}J\boldsymbol{r} + \xi_1\nabla(\boldsymbol{r}\wedge\nabla M) + \boldsymbol{\rho}_{tt} - (\boldsymbol{\rho}\cdot\nabla)\nabla M \\ = -\frac{\tau_{tt}}{2}BJ\boldsymbol{r} + \xi_{1t}B\boldsymbol{r} - BJ\boldsymbol{\rho}_t \end{aligned} \tag{9}$$

where $B = \nabla\wedge\boldsymbol{A}$. Once a solution of equations (8), (9) has been obtained, one simply integrates (6) and (7) to determine the corresponding function ϕ_1. (Recall that in one spatial dimension there is only one such compatibility condition [9].) In general, the compatibility conditions (8), (9) possess only the obvious solution

$$\tau_t = \xi_1 = \xi_0 = \eta_0 = 0 \tag{10}$$

† We adopt here the usual convention of the literature on symmetries of differential equations of denoting the dependent variable by u.

corresponding to the *trivial* symmetries

$$X_1 = \partial_t \qquad X_2 = u\partial_u \qquad X_3 = \phi_0\partial_u$$

which, of course, reflect the time-translational invariance and linearity of the Fokker–Planck equation. However, just as in one spatial dimension, the compatibility conditions admit nontrivial solutions provided there is a linear relationship between the coefficient functions of τ, ξ_1, ξ_0 and η_0 and their time derivatives [10, 11].

2. Irrotational case: $\nabla \wedge \boldsymbol{A} = 0$

We shall assume in this section that the drift vector $\boldsymbol{A}$ is irrotational, i.e. that B vanishes identically. The physical significance of this condition, known as the condition of *detailed balance*, has received ample study in the literature, [2, 3]. As a direct consequence, there exists a function A such that $\boldsymbol{A} = \nabla A$. Note that in this case the scale transformation $u \mapsto v = \mathrm{e}^{-A}u$ will map the Fokker–Planck equation (2) into the time-dependent Schrödinger equation with imaginary time given by

$$-v_t = -\tfrac{1}{2}\Delta v + Mv. \tag{11}$$

It follows immediately that two Fokker–Planck equations (2) with the same function M have isomorphic symmetry algebras. The RHS of equations (8) and (9) vanish identically, so in particular ξ_1 is a constant γ. We can then integrate equation (6), obtaining

$$\phi_1 = -\frac{\tau_{tt}}{4}r^2 + \frac{\tau_t}{2}\boldsymbol{r}\cdot\nabla A - \gamma\boldsymbol{r}\wedge\nabla A - \boldsymbol{\rho}_t\cdot\boldsymbol{r} + \boldsymbol{\rho}\cdot\nabla A + \varphi \tag{12}$$

where the function $\varphi(t)$ is determined up to a constant by equation (7) once the compatibility conditions have been fulfilled. Moreover, the LHS of the vector-valued equation (9) may be readily written as a gradient, leading to the single scalar equation

$$\omega + \frac{\tau_{ttt}}{4}r^2 - \frac{\tau_t}{2}(2 + \boldsymbol{r}\cdot\nabla)M + \gamma(\boldsymbol{r}\wedge\nabla M) + \boldsymbol{\rho}_{tt}\cdot\boldsymbol{r} - \boldsymbol{\rho}\cdot\nabla M = 0 \tag{13}$$

where $\omega(t)$ is an arbitrary function. Our purpose is to determine all possible linear relationships between the functions 1, r^2, $(2 + \boldsymbol{r}\cdot\nabla)M$, $\boldsymbol{r}\wedge\nabla M$, x, y, M_x and M_y which appear in equation (13). Each of the relationships may lead to a nontrivial symmetry of the Fokker–Planck equation. In order to obtain these relationships one needs to find the general solution of certain partial differential equations.

The most general function M consistent with $\tau_t \neq 0$ must satisfy the differential equation

$$(2 + \boldsymbol{r}\cdot\nabla)M = \lambda(\boldsymbol{r}\wedge\nabla M) + \mu M_x + \nu M_y + \lambda' r^2 + \mu' x + \nu' y + \sigma \tag{14}$$

for some real parameters λ, μ, ν, λ', μ', ν' and σ. The general solution of equation (14) is

$$M = C(\lambda\log\bar{r} + \bar{\theta})\bar{r}^{-2} + c\bar{r}^2 + a\bar{x} + b\bar{y} + c_0 \tag{15}$$

where C is an arbitrary function, and

$$\begin{aligned}
&\bar{x} = \bar{r}\cos\bar{\theta} = x + x_0 \qquad \bar{y} = \bar{r}\sin\bar{\theta} = y + y_0\\
&x_0 = \frac{\lambda\nu - \mu}{1 + \lambda^2} \qquad y_0 = -\frac{\lambda\mu + \nu}{1 + \lambda^2}\\
&c = \frac{\lambda'}{4} \qquad a = \frac{3\mu' + \lambda\nu' - 2\lambda'(\lambda y_0 + 3x_0)}{9 + \lambda^2} \qquad b = \frac{3\nu' - \lambda\mu' + 2\lambda'(\lambda x_0 - 3y_0)}{9 + \lambda^2}
\end{aligned} \tag{15$'$}$$

with c_0 an arbitrary constant. One can likewise determine the most general function M compatible with $\tau_t = 0$ and $\gamma \neq 0$ by solving the differential equation

$$\boldsymbol{r}\wedge\nabla M = \mu M_x + \nu M_y + \mu' x + \nu' y + \sigma \tag{16}$$

for real parameters μ, ν, μ', ν' and σ. The general solution of equation (16) is

$$M = C(\bar r) - \nu'\bar x + \mu'\bar y + d\bar\theta \tag{17}$$

where C is an arbitrary function, and

$$\bar x = \bar r\cos\bar\theta = x - \nu \qquad \bar y = \bar r\sin\bar\theta = y + \mu \qquad d = \mu'\nu - \mu\nu' + \sigma. \tag{17$'$}$$

Note that M in equation (17) reduces to the form (15) if the constant d vanishes and $C(\bar r)$ is a linear combination of $\bar r^2$, $\bar r^{-2}$ and 1. Finally, any function M consistent with $\tau_t = \gamma = 0$ and $\xi_0 \neq 0$ satisfies

$$M_x = \nu M_y + \mu' x + \nu' y + \sigma \tag{18}$$

for real parameters ν, μ', ν' and σ. The general solution of (18) is

$$M = C(\nu x + y) + \tfrac{1}{2}(\mu' + \nu\nu')x^2 + \nu' xy + \sigma x. \tag{19}$$

The form of the Fokker–Planck equation (2) is invariant under rotations, translations and dilatations of the form $(\boldsymbol{r}, t) \mapsto (\lambda\boldsymbol{r}, \lambda^2 t)$, $\boldsymbol{A} \mapsto \lambda^{-1}\boldsymbol{A}$. Since these operations do not modify the symmetry properties of the equation, we shall assume without any loss of generality that $x_0 = y_0 = 0$ in equations (15$'$) and $\mu = \nu = 0$ in (17$'$), so $\bar x = x$, $\bar y = y$, $\bar r = r$ and $\bar\theta = \theta$. In order to determine explicitly nontrivial solutions for τ, ξ_0, η_0 and $\xi_1 = \gamma$, we substitute the expressions (15), (17) and (19) for M into (13), and impose that the time-dependent coefficients of each of the linearly independent functions of the spatial variables vanish identically. In so doing, one needs to consider new possible linear relations for some particular forms of the arbitrary function C appearing in M. The function φ in equation (12) can then be determined up to a constant from equation (7). Finally, the corresponding nontrivial symmetry generators are obtained by substituting equations (4), (5) and (12) into (3):

$$\begin{aligned} X = {} & \left(\frac{\tau_t}{2}x + \gamma y + \xi_0\right)\partial_x + \left(\frac{\tau_t}{2}y - \gamma x + \eta_0\right)\partial_y + \tau\partial_t \\ & + \left(-\frac{\tau_{tt}}{4}r^2 + \frac{\tau_t}{2}\boldsymbol{r}\cdot\nabla A - \gamma\boldsymbol{r}\wedge\nabla A - \boldsymbol{\rho}_t\cdot\boldsymbol{r} + \boldsymbol{\rho}\cdot\nabla A + \varphi\right)u\partial_u \end{aligned} \tag{20}$$

where $\boldsymbol{\rho} = (\xi_0, \eta_0)$ and A satisfies the nonlinear PDE

$$2M = (\nabla A)^2 + \Delta A. \tag{21}$$

In what follows we provide the complete list of functions M for which the Fokker–Planck equation (2) admits nontrivial symmetries, up to possible translations, rotations and dilatations. For each such function M we give the form of τ, ξ_0, η_0, γ and φ, from which the reader can immediately obtain the associated nontrivial symmetry generators from equation (20) by giving suitable values to the group parameters α_i, β_i, γ and δ_i. We also denote by s the maximum number of nontrivial symmetry generators. Note that in general the function A cannot be determined explicitly from M—a step that would require solving the nonlinear PDE (21). However, given *any* Fokker–Planck equation (2) with an irrotational drift vector $\boldsymbol{A} = \nabla A$, it may be readily verified whether the corresponding function M belongs to the list.

Case 1.1a: $M = \frac{C_0}{x^2} + by + c_0$; $C_0 \neq 0$, $(s = 4)$.

$$\tau = \delta_2 t^2 + \delta_1 t \qquad \gamma = \xi_0 = 0 \qquad \eta_0 = \frac{b\delta_2}{2}t^3 + \frac{3b\delta_1}{4}t^2 + \beta_1 t + \beta_0$$

$$\varphi = -\frac{b^2\delta_2}{8}t^4 - \frac{b^2\delta_1}{4}t^3 - \left(\frac{b\beta_1}{2} + c_0\delta_2\right)t^2 - (\delta_2 + c_0\delta_1 + b\beta_0)t.$$

Case 1.1b:† $M = \frac{C_0}{x^2} + cr^2 + by + c_0$; $C_0 \neq 0$, $(s = 4)$.

$$\tau = \delta_1 e^{2\sqrt{2c}t} + \delta_2 e^{-2\sqrt{2c}t} \qquad \gamma = \xi_0 = 0$$

$$\eta_0 = \frac{b\delta_1}{\sqrt{2c}} e^{2\sqrt{2c}t} + \frac{b\delta_2}{\sqrt{2c}} e^{-2\sqrt{2c}t} + \beta_1 e^{\sqrt{2c}t} + \beta_2 e^{-\sqrt{2c}t}$$

$$\varphi = -\left(\sqrt{2c} + c_0 + \frac{b^2}{4c}\right)\delta_1 e^{2\sqrt{2c}t} + \left(\sqrt{2c} - c_0 - \frac{b^2}{4c}\right)\delta_2 e^{-2\sqrt{2c}t} - \frac{b\beta_1}{\sqrt{2c}} e^{\sqrt{2c}t} + \frac{b\beta_2}{\sqrt{2c}} e^{-\sqrt{2c}t}.$$

Case 1.2a: $M = C(\theta)r^{-2} + c_0$, $(s = 2)$.

$$\tau = \delta_2 t^2 + \delta_1 t \qquad \gamma = \xi_0 = \eta_0 = 0 \qquad \varphi = -c_0\delta_2 t^2 - (\delta_2 + c_0\delta_1)t$$

with $C(\theta) \neq (C_0 \cos\theta + C_1 \sin\theta)^{-2}$ and $C' \neq 0$.

Case 1.2b: $M = C(\theta)r^{-2} + cr^2 + c_0$, $(s = 2)$.

$$\tau = \delta_1 e^{2\sqrt{2c}t} + \delta_2 e^{-2\sqrt{2c}t} \qquad \gamma = \xi_0 = \eta_0 = 0$$

$$\varphi = -\left(\sqrt{2c} + c_0\right)\delta_1 e^{2\sqrt{2c}t} + \left(\sqrt{2c} - c_0\right)\delta_2 e^{-2\sqrt{2c}t}$$

with $C(\theta) \neq (C_0 \cos\theta + C_1 \sin\theta)^{-2}$ and $C' \neq 0$.

Case 1.3: $M = C(\lambda \log r + \theta)r^{-2} + c_0$; $C' \neq 0 \neq \lambda$, $(s = 1)$.

$$\tau = \frac{2\gamma}{\lambda} t \qquad \xi_0 = \eta_0 = 0 \qquad \varphi = -\frac{2c_0\gamma}{\lambda} t.$$

Case 1.4a: $M = C_0 r^{-2} + ax + by + c_0$; $C_0 \neq 0$, $(s = 3)$.

$$\tau = \delta_2 t^2 + \delta_1 t \qquad \xi_0 = \eta_0 = 0 \qquad \varphi = -c_0\delta_2 t^2 - (\delta_2 + c_0\delta_1)t$$

with the constraint $\delta_1 = \delta_2 = 0$ if $a \neq 0$ or $b \neq 0$.

Case 1.4b: $M = C_0 r^{-2} + cr^2 + ax + by + c_0$; $C_0 \neq 0$, $(s = 3)$.

$$\tau = \delta_1 e^{2\sqrt{2c}t} + \delta_2 e^{-2\sqrt{2c}t} \qquad \xi_0 = \eta_0 = 0$$

$$\varphi = -\left(\sqrt{2c} + c_0\right)\delta_1 e^{2\sqrt{2c}t} + \left(\sqrt{2c} - c_0\right)\delta_2 e^{-2\sqrt{2c}t}$$

with the constraint $\delta_1 = \delta_2 = 0$ if $a \neq 0$ or $b \neq 0$.

Case 1.5a: $M = ax + by + c_0$, $(s = 7)$.

$$\tau = \delta_2 t^2 + \delta_1 t$$

$$\xi_0 = \frac{a\delta_2}{2} t^3 + \frac{1}{4}(3a\delta_1 - 2b\gamma)t^2 + \alpha_1 t + \alpha_0$$

$$\eta_0 = \frac{b\delta_2}{2} t^3 + \frac{1}{4}(3b\delta_1 + 2a\gamma)t^2 + \beta_1 t + \beta_0$$

$$\varphi = -\tfrac{1}{8}(a^2 + b^2)\delta_2 t^4 - \tfrac{1}{4}(a^2 + b^2)\delta_1 t^3 - (\tfrac{1}{2}(a\alpha_1 + b\beta_1) + c_0\delta_2)t^2 - (\delta_2 + c_0\delta_1 + a\alpha_0 + b\beta_0)t.$$

† The parameter c which appears in cases b and case 1.7a is assumed to be nonzero.

Case 1.5b: $M = cr^2 + ax + by + c_0$, $(s = 7)$.

$$\tau = \delta_1 \mathrm{e}^{2\sqrt{2c}t} + \delta_2 \mathrm{e}^{-2\sqrt{2c}t}$$

$$\xi_0 = \frac{a\delta_1}{\sqrt{2c}}\mathrm{e}^{2\sqrt{2c}t} + \frac{a\delta_2}{\sqrt{2c}}\mathrm{e}^{-2\sqrt{2c}t} + \alpha_1 \mathrm{e}^{\sqrt{2c}t} + \alpha_2 \mathrm{e}^{-\sqrt{2c}t} + \frac{b\gamma}{2c}$$

$$\eta_0 = \frac{b\delta_1}{\sqrt{2c}}\mathrm{e}^{2\sqrt{2c}t} + \frac{b\delta_2}{\sqrt{2c}}\mathrm{e}^{-2\sqrt{2c}t} + \beta_1 \mathrm{e}^{\sqrt{2c}t} + \beta_2 \mathrm{e}^{-\sqrt{2c}t} - \frac{a\gamma}{2c}$$

$$\varphi = -\left(\sqrt{2c} + c_0 + \frac{a^2+b^2}{4c}\right)\delta_1 \mathrm{e}^{2\sqrt{2c}t} + \left(\sqrt{2c} - c_0 - \frac{a^2+b^2}{4c}\right)\delta_2 \mathrm{e}^{-2\sqrt{2c}t}$$
$$\qquad - \frac{a\alpha_1 + b\beta_1}{\sqrt{2c}}\mathrm{e}^{\sqrt{2c}t} + \frac{a\alpha_2 + b\beta_2}{\sqrt{2c}}\mathrm{e}^{-\sqrt{2c}t}.$$

Case 1.6: $M = C(r) + d\theta$, $(s = 1)$.

$$\tau = \xi_0 = \eta_0 = 0 \qquad \varphi = d\gamma t$$

with $C(r) \neq C_0 r^{-2} + C_1 r^2 + c_0$ if $d = 0$.

Case 1.7a: $M = cx^2 + ax + by + c_0$, $(s = 4)$.

$$\tau = \gamma = 0 \qquad \xi_0 = \alpha_1 \mathrm{e}^{\sqrt{2c}t} + \alpha_2 \mathrm{e}^{-\sqrt{2c}t} \qquad \eta_0 = \beta_1 t + \beta_0$$
$$\varphi = -\frac{a\alpha_1}{\sqrt{2c}}\mathrm{e}^{\sqrt{2c}t} + \frac{a\alpha_2}{\sqrt{2c}}\mathrm{e}^{-\sqrt{2c}t} - \frac{b\beta_1}{2}t^2 - b\beta_0 t.$$

Case 1.7b: $M = c(x^2 - y^2) + ax + by + c_0$, $(s = 4)$.

$$\tau = \gamma = 0 \qquad \xi_0 = \alpha_1 \mathrm{e}^{\sqrt{2c}t} + \alpha_2 \mathrm{e}^{-\sqrt{2c}t} \qquad \eta_0 = \beta_1 \mathrm{e}^{\sqrt{-2c}t} + \beta_2 \mathrm{e}^{-\sqrt{-2c}t}$$
$$\varphi = -\frac{a\alpha_1}{\sqrt{2c}}\mathrm{e}^{\sqrt{2c}t} + \frac{a\alpha_2}{\sqrt{2c}}\mathrm{e}^{-\sqrt{2c}t} - \frac{b\beta_1}{\sqrt{-2c}}\mathrm{e}^{\sqrt{-2c}t} + \frac{b\beta_2}{\sqrt{-2c}}\mathrm{e}^{-\sqrt{-2c}t}.$$

Case 1.8a: $M = C(x) + by$, $(s = 2)$.

$$\tau = \gamma = \xi_0 = 0 \qquad \eta_0 = \beta_1 t + \beta_0 \qquad \varphi = -\frac{b\beta_1}{2}t^2 - b\beta_0 t$$

with $C_0 x^2 + ax + c_0 \neq C(x) \neq \frac{C_0}{x^2} + c_0$.

Case 1.8b: $M = C(x) + cy^2 + by$, $(s = 2)$.

$$\tau = \gamma = \xi_0 = 0 \qquad \eta_0 = \beta_1 \mathrm{e}^{\sqrt{2c}t} + \beta_2 \mathrm{e}^{-\sqrt{2c}t}$$
$$\varphi = -\frac{b\beta_1}{\sqrt{2c}}\mathrm{e}^{\sqrt{2c}t} + \frac{b\beta_2}{\sqrt{2c}}\mathrm{e}^{-\sqrt{2c}t}$$

with $C_0 x^2 + ax + c_0 \neq C(x) \neq \frac{C_0}{x^2} + cx^2 + c_0$.

The list above could be further reduced† by using point transformations preserving the form of the Fokker–Planck equation (2) involving the dependent variable u. Although no attempt will be made here to determine the complete group of point transformations leaving the form of equation (2) invariant, we can take advantage in the irrotational case of what is known for

† We prefer to leave the list in its present form so that the reader may check if a given Fokker–Planck equation (2) possesses nontrivial symmetries without the need to perform complicated transformations.

the time-dependent Schrödinger equation. In one spatial dimension, the group of projectable point transformations preserving the form of the time-dependent Schrödinger equation was essentially obtained by Ray [13]. A straightforward generalization of Ray's transformations was subsequently applied by Kaushal to the time-dependent Schrödinger equation in 2 + 1 dimensions [14]. Closely following Kaushal, it may be easily verified that the transformation

$$\begin{aligned}
&\bar{\boldsymbol{r}} = \frac{1}{f(t)}\boldsymbol{r} + \boldsymbol{g} \qquad \bar{t} = \int^t \frac{\mathrm{d}s}{f^2(s)} \qquad \boldsymbol{g} = (g_1(t), g_2(t)) \\
&\bar{v}(\bar{\boldsymbol{r}}, \bar{t}) = \exp\left[\frac{f'}{2f}r^2 - f\boldsymbol{g}'\cdot\boldsymbol{r} + h(t)\right]v(\boldsymbol{r}, t)
\end{aligned} \tag{22}$$

maps the time-dependent Schrödinger equation (11) into the time-dependent Schrödinger equation

$$-\bar{v}_{\bar{t}} = -\tfrac{1}{2}\bar{\Delta}\bar{v} + \bar{M}\bar{v}$$

with potential

$$\bar{M} = f^2 M - \frac{ff''}{2}r^2 + f^2(f\boldsymbol{g}'' + 2f'\boldsymbol{g}')\cdot\boldsymbol{r} - \frac{f^4}{2}(\boldsymbol{g}')^2 + ff' - f^2 h'. \tag{23}$$

In general, the transformation (22), (23) maps a time-independent potential into a time-dependent one, unless f, $\boldsymbol{g}$ and h are constants. However, for some particular time-independent potentials and transformations, the transformed potential is also time independent. Thus, if the functions M and $\bar{M}$ associated to a pair of Fokker–Planck equations (2) with irrotational drift vectors are related by such a transformation, then the equations themselves are related by a point transformation, and their symmetry groups are therefore isomorphic. This is precisely what happens in our list among each a and b subcases. Indeed, if

$$f^3 f'' = -2c \qquad \boldsymbol{g} = 0 \qquad h' = f'/f \qquad \text{i.e.} \qquad f^2 = 2\sqrt{2c}t \qquad h = \log f$$

the function M of each a subcase is transformed into the function M of the corresponding b subcase. Moreover, the functions M in cases 1.5 are equivalent under suitable transformations of the form (22) to $M = 0$, so their corresponding Fokker–Planck equations may be reduced to the heat equation.

3. Generic case: $\nabla \wedge \boldsymbol{A} \not\equiv \boldsymbol{0}$

In this section we shall classify all Fokker–Planck equations (2) (with $\beta = 1$) possessing nontrivial symmetries in the case in which the rotational B does not vanish identically. In the first step one needs to find the functions B for which the first compatibility condition (8) admits solutions different from the trivial solution (10). Just as in the previous section, each linear relationship between 1, $(2 + \boldsymbol{r}\cdot\nabla)B$, $\boldsymbol{r}\wedge\nabla B$, B_x and B_y may lead to a nontrivial symmetry. Note that these linear relationships are in fact first-order linear PDE's for the rotational B which can be solved in closed form. The explicit forms of the functions B satisfying one of these linear relationships, up to possible translations, rotations, and dilatations of the form $(\boldsymbol{r}, t) \mapsto (\lambda\boldsymbol{r}, \lambda^2 t)$, $\boldsymbol{A} \mapsto \lambda^{-1}\boldsymbol{A}$, are:

$$\begin{aligned}
&B = \frac{\hat{C}_0}{x^2} + \hat{c}_0 \qquad B = \frac{\hat{C}_0}{r^2} + \hat{c}_0 \qquad B = \hat{c}_0 \qquad B = \frac{\hat{C}(\theta)}{r^2} \\
&B = \frac{\hat{C}(\lambda\log r + \theta)}{r^2} + \hat{c}_0 \qquad B = \hat{C}(r) + 2\hat{d}\theta \qquad B = \hat{C}(x).
\end{aligned}$$

Here we assume that $\hat{C}_0 \neq 0$ and $\hat{C}' \neq 0$. In the second step one replaces each of these functions B and the resulting form of ξ_1, ξ_0, η_0 and τ into the vector-valued compatibility condition (9) in order to determine the functions M for which the final form of ξ_1, ξ_0, η_0 and τ is nontrivial.

We next present the exhaustive list of functions $B \neq 0$ and M for which the Fokker–Planck equation (2) possesses nontrivial symmetries. We provide in each case the explicit form of the functions ξ_1, ξ_0, η_0, τ, and ϕ_1 determining the symmetry generators via the formula†

$$X = \left(\frac{\tau_t}{2}x + \xi_1 y + \xi_0\right)\partial_x + \left(\frac{\tau_t}{2}y - \xi_1 x + \eta_0\right)\partial_y + \tau\partial_t + \phi_1 u\partial_u. \qquad (24)$$

Case 2.1a: $B = \frac{\hat{C}_0}{x^2} + \hat{c}_0$, $M = C(x) + cy$, $(s = 1)$.

$$\tau = \xi_1 = \xi_0 = 0 \qquad \eta_0 = \beta_0 \qquad \phi_1 = \beta_0\left(A_2 + \frac{\hat{C}_0}{x} - \hat{c}_0 x - ct\right).$$

Case 2.1b: $B = \frac{\hat{C}_0}{x^2} + \hat{c}_0$, $M = C(x) + c(\hat{c}_0 x - \frac{\hat{C}_0}{x})y + \frac{c^2}{2}y^2$, $(s = 1)$.

$$\tau = \xi_1 = \xi_0 = 0 \qquad \eta_0 = \beta_1 \mathrm{e}^{ct} \qquad \phi_1 = \beta_1 \mathrm{e}^{ct}\left(A_2 + \frac{\hat{C}_0}{x} - \hat{c}_0 x - cy\right).$$

Case 2.2a: $B = \frac{\hat{C}_0}{x^2}$, $M = \frac{C_0}{x^2} + \frac{c}{x} + c_0$, $(s = 2)$.

$$\tau = \delta_1 t \qquad \xi_1 = \xi_0 = 0 \qquad \eta_0 = -\frac{c\delta_1}{2\hat{C}_0}t + \beta_0$$

$$\phi_1 = \frac{\delta_1}{2}\left(\boldsymbol{r}\cdot\boldsymbol{A} + \frac{\hat{C}_0 y}{x} + \frac{cy}{\hat{C}_0} - \left(\frac{cA_2}{\hat{C}_0} + \frac{c}{x} + 2c_0\right)t\right) + \beta_0\left(A_2 + \frac{\hat{C}_0}{x}\right).$$

Case 2.2b: $B = \frac{\hat{C}_0}{x^2}$, $M = \frac{C_0}{x^2} + \frac{a}{x} + c(\frac{\hat{C}_0}{x} + \frac{a}{\hat{C}_0})y + \frac{c^2}{2}r^2 + c_0$, $(s = 2)$.

$$\tau = \delta_1 \mathrm{e}^{-2ct} \qquad \xi_1 = \xi_0 = 0 \qquad \eta_0 = \beta_1 \mathrm{e}^{-ct} - \frac{a\delta_1}{\hat{C}_0}\mathrm{e}^{-2ct}$$

$$\phi_1 = -\delta_1\left(c\boldsymbol{r}\cdot\boldsymbol{A} + c^2r^2 + \frac{c\hat{C}_0 y}{x} + \frac{2acy}{\hat{C}_0} + \frac{a}{\hat{C}_0}A_2 + \frac{a}{x} + c_0 + \frac{a^2}{2c\hat{C}_0^2} - c\right)\mathrm{e}^{-2ct}$$
$$+\beta_1\left(A_2 + cy + \frac{\hat{C}_0}{x} + \frac{a}{c\hat{C}_0}\right)\mathrm{e}^{-ct}.$$

Case 3.1a: $B = \frac{\hat{C}_0}{r^2}$, $M = C(\lambda\log r + \theta) + c_0$; $C' \neq 0$, $(s = 1)$.

$$\tau = \delta_1 t \qquad \xi_1 = \frac{\lambda\delta_1}{2} \qquad \xi_0 = \eta_0 = 0$$

$$\phi_1 = \frac{\delta_1}{2}(\boldsymbol{r}\cdot\boldsymbol{A} + \hat{C}_0\theta + \lambda(\hat{C}_0\log r - \boldsymbol{r}\wedge\boldsymbol{A}) - 2c_0 t).$$

† The general form of the function $\phi_1(x, y, t)$ can be found provided the first compatibility condition (8) holds. In practice, however, it is preferable to give a simpler expression for each particular case.

Case 3.1b: $B = \frac{\hat{C}_0}{r^2} + \hat{c}_0$, $M = C(\frac{\hat{c}_0\hat{C}_0}{c}\log r + \theta)r^{-2} + (\frac{c^2}{\hat{C}_0^2} + \hat{c}_0^2)\frac{r^2}{8} + \frac{\hat{c}_0\hat{C}_0}{2}\log r + \frac{c}{2}\theta + c_0$, $(s = 1)$.

$$\tau = \delta_1 \mathrm{e}^{-c/\hat{C}_0 t} \qquad \xi_1 = -\frac{\hat{c}_0\delta_1}{2}\mathrm{e}^{-c/\hat{C}_0 t} \qquad \xi_0 = \eta_0 = 0$$

$$\phi_1 = -\frac{\delta_1}{2}\mathrm{e}^{-c/\hat{C}_0 t}\left(\frac{c}{\hat{C}_0}\boldsymbol{r}\cdot\boldsymbol{A} + \left(\frac{c^2}{\hat{C}_0^2} + \hat{c}_0^2\right)\frac{r^2}{2} + \hat{c}_0(\hat{C}_0\log r - \boldsymbol{r}\wedge\boldsymbol{A}) + c\theta + 2c_0 - \frac{c}{\hat{C}_0}\right)$$

with $c \neq 0$.

Case 3.2a: $B = \frac{\hat{C}_0}{r^2} + \hat{c}_0$, $M = C_0 r^{-2} + \frac{\hat{c}_0^2}{2}r^2 + \frac{\hat{c}_0\hat{C}_0}{2}\log r + c_0$, $(s = 2)$.

$$\tau = \delta_1 t \qquad \xi_1 = -\frac{\hat{c}_0\delta_1}{2}t + \gamma \qquad \xi_0 = \eta_0 = 0$$

$$\phi_1 = \frac{\delta_1}{2}(\boldsymbol{r}\cdot\boldsymbol{A} + \hat{C}_0\theta) + \left(\frac{\hat{c}_0\delta_1 t}{2} - \gamma\right)\left(\boldsymbol{r}\wedge\boldsymbol{A} - \hat{C}_0\log r - \frac{\hat{c}_0 r^2}{2}\right) - c_0\delta_1 t.$$

Case 3.2b: $B = \frac{\hat{C}_0}{r^2} + \hat{c}_0$, $M = C_0 r^{-2} + (\frac{c^2}{\hat{C}_0^2} + \hat{c}_0^2)\frac{r^2}{8} + \frac{\hat{c}_0\hat{C}_0}{2}\log r + \frac{c}{2}\theta + c_0$, $(s = 2)$.

$$\tau = \delta_1 \mathrm{e}^{-c/\hat{C}_0 t} \qquad \xi_1 = -\frac{\hat{c}_0\delta_1}{2}\mathrm{e}^{-c/\hat{C}_0 t} + \gamma \qquad \xi_0 = \eta_0 = 0$$

$$\phi_1 = -\frac{\delta_1}{2}\mathrm{e}^{-c/\hat{C}_0 t}\left(\frac{c}{\hat{C}_0}\boldsymbol{r}\cdot\boldsymbol{A} + c\theta + 2c_0 - \frac{c}{\hat{C}_0}\right) + \left(\gamma\hat{c}_0 - \frac{\delta_1}{2}\mathrm{e}^{-c/\hat{C}_0 t}\left(\frac{c^2}{\hat{C}_0^2} + \hat{c}_0^2\right)\right)\frac{r^2}{2}$$
$$+ \left(\frac{\hat{c}_0\delta_1}{2}\mathrm{e}^{-c/\hat{C}_0 t} - \gamma\right)(\boldsymbol{r}\wedge\boldsymbol{A} - \hat{C}_0\log r)$$

with $c \neq 0$.

Case 3.3: $B = \frac{\hat{C}_0}{r^2} + \hat{c}_0$, $M = C(r) + d\theta$, $(s = 1)$.

$$\tau = \xi_0 = \eta_0 = 0 \qquad \xi_1 = \gamma \qquad \phi_1 = \gamma\left(\hat{C}_0\log r + \frac{\hat{c}_0}{2}r^2 - \boldsymbol{r}\wedge\boldsymbol{A} + dt\right)$$

with $C(r) \neq C_0 r^{-2} + (\frac{d^2}{\hat{C}_0^2} + \frac{\hat{c}_0^2}{4})\frac{r^2}{2} + \frac{\hat{c}_0\hat{C}_0}{2}\log r + c_0$.

Case 4.1: $B = \hat{c}_0$, $M = C(\lambda\log r + \theta)r^{-2} + \hat{c}_0^2(1 + \lambda^{-2})\frac{r^2}{8} + c_0$, $(s = 1)$.

$$\tau = \delta_1 \mathrm{e}^{-\hat{c}_0/\lambda t} \qquad \xi_1 = -\frac{\hat{c}_0\delta_1}{2}\mathrm{e}^{-\hat{c}_0/\lambda t}, \qquad \xi_0 = \eta_0 = 0$$

$$\phi_1 = -\frac{\delta_1\hat{c}_0}{2}\mathrm{e}^{-\hat{c}_0/\lambda t}\left(\lambda^{-1}\boldsymbol{r}\cdot\boldsymbol{A} + \hat{c}_0(1 + \lambda^{-2})\frac{r^2}{2} - \boldsymbol{r}\wedge\boldsymbol{A} + \frac{2c_0}{\hat{c}_0} - \lambda^{-1}\right)$$

with $C' \neq 0 \neq \lambda$.

Case 4.2a: $B = \hat{c}_0$, $M = C_0 r^{-2} + \frac{\hat{c}_0^2}{8}r^2 + c_0$, $(s = 3)$.

$$\tau = \delta_2 t^2 + \delta_1 t \qquad \xi_1 = -\frac{\hat{c}_0}{2}(\delta_2 t^2 + \delta_1 t) + \gamma \qquad \xi_0 = \eta_0 = 0$$

$$\phi_1 = \left(\frac{\hat{c}_0}{2}(\delta_2 t^2 + \delta_1 t) - \gamma\right)\left(\boldsymbol{r}\wedge\boldsymbol{A} - \frac{\hat{c}_0}{2}r^2\right) - \frac{\delta_2}{2}r^2 - \frac{c_0\delta_1}{2}t + \left(\delta_2 t + \frac{\delta_1}{2}\right)(\boldsymbol{r}\cdot\boldsymbol{A} - c_0 t).$$

Case 4.2b: $B = \hat{c}_0$, $M = C_0 r^{-2} + \frac{c^2+\hat{c}_0^2}{8} r^2 + c_0$, $(s = 3)$.

$$\tau = \delta_1 \mathrm{e}^{ct} + \delta_2 \mathrm{e}^{-ct} \qquad \xi_1 = -\frac{\hat{c}_0}{2}(\delta_1 \mathrm{e}^{ct} + \delta_2 \mathrm{e}^{-ct}) + \gamma \qquad \xi_0 = \eta_0 = 0$$

$$\begin{aligned}\phi_1 = (\delta_1 \mathrm{e}^{ct} + \delta_2 \mathrm{e}^{-ct}) \left(\frac{\hat{c}_0}{2} \boldsymbol{r} \wedge \boldsymbol{A} - \frac{c^2 + \hat{c}_0}{4} r^2 - c_0 \right) + \gamma \left(\frac{\hat{c}_0}{2} r^2 - \boldsymbol{r} \wedge \boldsymbol{A} \right) \\ + \frac{c}{2}(\delta_1 \mathrm{e}^{ct} - \delta_2 \mathrm{e}^{-ct})(\boldsymbol{r} \cdot \boldsymbol{A} - 1).\end{aligned}$$

Case 4.3: $B = \hat{c}_0$, $M = \frac{c^2+\hat{c}_0^2}{8} r^2 + ax + by + c_0$, $(s = 7)$.

The functions τ and ξ_1 are given as in case 4.2a (case 4.2b) if $c = 0$ $(c \neq 0)$. The functions ξ_0 and η_0 are the general solution of the linear second-order system with constant coefficients given by†

$$\begin{aligned}\xi_{0tt} - \frac{c^2 + \hat{c}_0^2}{2} \xi_0 + \hat{c}_0 \eta_{0t} &= \frac{3a}{2} \tau_t + b \left(\frac{\hat{c}_0}{2} \tau - \gamma \right) \\ \eta_{0tt} - \frac{c^2 + \hat{c}_0^2}{2} \eta_0 - \hat{c}_0 \xi_{0t} &= \frac{3b}{2} \tau_t - a \left(\frac{\hat{c}_0}{2} \tau - \gamma \right).\end{aligned} \tag{25}$$

The function ϕ_1 in (24) is then computed from equations (6) and (7) as usual.

Case 4.4: $B = \hat{c}_0$, $M = C(r) + d\theta$, $(s = 1)$.

$$\tau = \xi_0 = \eta_0 = 0 \qquad \xi_1 = \gamma \qquad \phi_1 = \gamma \left(\frac{\hat{c}_0}{2} r^2 - \boldsymbol{r} \wedge \boldsymbol{A} + dt \right)$$

with $C(r) \neq C_0 r^{-2} + C_1 r^2 + c_0$ if $d = 0$.

Case 4.5a: $B = \hat{c}_0$, $M = \frac{\hat{c}_0^2}{2} x^2 + ax + by + c_0$, $(s = 4)$.

$$\tau = \xi_1 = 0 \qquad \xi_0 = \alpha_2 t^2 + \alpha_1 t + \alpha_0 \qquad \eta_0 = \frac{\hat{c}_0 \alpha_2}{3} t^3 + \frac{\hat{c}_0 \alpha_1}{2} t^2 + \left(\hat{c}_0 \alpha_0 - \frac{2\alpha_2}{\hat{c}_0} \right) t + \beta_0$$

$$\begin{aligned}\phi_1 = \boldsymbol{\rho} \cdot \boldsymbol{A} - \left(\hat{c}_0^2 \left(\frac{\alpha_2}{3} t^3 + \frac{\alpha_1}{2} t^2 + \alpha_0 t \right) + \beta_0 \hat{c}_0 + \alpha_1 \right) x \\ + \frac{2\alpha_2}{\hat{c}_0} y - \left(\hat{c}_0 \alpha_2 b \frac{t^4}{12} + (\hat{c}_0 \alpha_1 b + 2\alpha_2 a) \frac{t^3}{6} \right. \\ \left. + \left(\hat{c}_0 \alpha_0 b - \frac{2\alpha_2 b}{\hat{c}_0} + \alpha_1 a \right) \frac{t^2}{2} + (\beta_0 b + a_0 a) t \right).\end{aligned}$$

Case 4.5b: $B = \hat{c}_0$, $M = \frac{c^2+\hat{c}_0^2}{2} x^2 + ax + by + c_0$, $(s = 4)$.

$$\tau = \xi_1 = 0 \qquad \xi_0 = \alpha_1 \mathrm{e}^{ct} + \alpha_2 \mathrm{e}^{-ct} + \alpha_3$$

$$\eta_0 = \frac{\hat{c}_0 \alpha_1}{c} \mathrm{e}^{ct} - \frac{\hat{c}_0 \alpha_2}{c} \mathrm{e}^{-ct} + \frac{(c^2 + \hat{c}_0^2) \alpha_3}{\hat{c}_0} t + \beta_0$$

$$\begin{aligned}\phi_1 = \boldsymbol{\rho} \cdot \boldsymbol{A} - \left(\left(\frac{\alpha_1}{c} \mathrm{e}^{ct} - \frac{\alpha_2}{c} \mathrm{e}^{-ct} + \alpha_3 t \right) (c^2 + \hat{c}_0^2) + \hat{c}_0 \beta_0 \right) x - \frac{\alpha_3 c}{\hat{c}_0} y \\ - \left(\frac{\alpha_1}{c^2} (ac + b\hat{c}_0) \mathrm{e}^{ct} + \frac{\alpha_2}{c^2} (-ac + b\hat{c}_0) \mathrm{e}^{-ct} + \frac{\alpha_3}{2} \left(a + \frac{b(c^2 + \hat{c}_0^2)}{\hat{c}_0} \right) t^2 + \beta_0 b t \right).\end{aligned}$$

† The resulting expressions of ξ_0 and η_0 are too cumbersome to be displayed here.

Case 4.6: $B = \hat{c}_0$, $M = c(x^2 - y^2) + ax + by + c_0$, $(s = 4)$.

$$\tau = \xi_1 = 0 \qquad \xi_0 = \alpha_1 e^{\lambda_+ t} + \alpha_2 e^{-\lambda_+ t} + \alpha_3 e^{\lambda_- t} + \alpha_4 e^{-\lambda_- t}$$

$$\eta_0 = \frac{2c - \lambda_+^2}{\hat{c}_0 \lambda_+} \alpha_1 e^{\lambda_+ t} - \frac{2c - \lambda_+^2}{\hat{c}_0 \lambda_+} \alpha_2 e^{-\lambda_+ t} + \frac{2c - \lambda_-^2}{\hat{c}_0 \lambda_-} \alpha_3 e^{\lambda_- t} - \frac{2c - \lambda_-^2}{\hat{c}_0 \lambda_-} \alpha_4 e^{-\lambda_- t}$$

$$\phi_1 = \boldsymbol{\rho} \cdot \boldsymbol{A} - (\xi_{0t} + \hat{c}_0 \eta_0) x + (\hat{c}_0 \xi_0 - \eta_{0t}) y - a \int^t \xi_0 - b \int^t \eta_0$$

where $\lambda_\pm = \frac{1}{2}\left(-\hat{c}_0^2 \pm \sqrt{\hat{c}_0^4 + 16c^4}\right)$.

Case 4.7a: $B = \hat{c}_0$, $M = C(x) + by$, $(s = 1)$.

$$\tau = \xi_1 = \xi_0 = 0 \qquad \eta_0 = \beta_0 \qquad \phi_1 = \beta_0 (A_2 - \hat{c}_0 x - bt)$$

with $C(x) \neq C_0 x^2 + ax + c_0$.

Case 4.7b: $B = \hat{c}_0$, $M = C(x) + \frac{c^2}{2} y^2 + c\hat{c}_0 xy + by$, $(s = 1)$.

$$\tau = \xi_1 = \xi_0 = 0 \qquad \eta_0 = \beta_1 e^{ct} \qquad \phi_1 = \beta_1 e^{ct}\left(A_2 - \hat{c}_0 x - cy - \frac{b}{c}\right)$$

with $C(x) \neq C_0 x^2 + ax + c_0$.

Case 5.1a: $B = \frac{\hat{C}(\theta)}{r^2}$, $M = \frac{C(\theta)}{r^2}$, $(s = 1)$.

$$\tau = \delta_1 t \qquad \xi_1 = \xi_0 = \eta_0 = 0 \qquad \phi_1 = \frac{\delta_1}{2}\left(\boldsymbol{r} \cdot \boldsymbol{A} + \int^\theta \hat{C}(\theta)\right)$$

with $\hat{C}(\theta) \neq (C_0 \cos\theta + C_1 \sin\theta)^{-2}$ and $\hat{C}' \neq 0$.

Case 5.1b: $B = \frac{\hat{C}(\theta)}{r^2}$, $M = \frac{C(\theta)}{r^2} + \frac{c^2}{8} r^2 + \frac{c}{2} \int^\theta \hat{C}(\theta)$, $(s = 1)$.

$$\tau = \delta_1 e^{-ct} \qquad \xi_1 = \xi_0 = \eta_0 = 0 \qquad \phi_1 = -\frac{\delta_1 c}{2} e^{-ct}\left(\boldsymbol{r} \cdot \boldsymbol{A} + \frac{c}{2} r^2 - 1 + \int^\theta \hat{C}(\theta)\right)$$

with $\hat{C}(\theta) \neq (C_0 \cos\theta + C_1 \sin\theta)^{-2}$ and $\hat{C}' \neq 0$.

Case 6.1a: $B = \frac{\hat{C}(\lambda \log r + \theta)}{r^2}$, $M = \frac{C(\lambda \log r + \theta)}{r^2} + c_0$, $(s = 1)$.

$$\tau = \delta_1 t \qquad \xi_1 = \frac{\lambda \delta_1}{2} \qquad \xi_0 = \eta_0 = 0$$

$$\phi_1 = \frac{\delta_1}{2}\left(\boldsymbol{r} \cdot \boldsymbol{A} - \lambda \boldsymbol{r} \wedge \boldsymbol{A} + \int^{\lambda \log r + \theta} \hat{C}(s)\, ds - 2c_0 t\right).$$

Case 6.1b: $B = \frac{\hat{C}(\lambda \log r + \theta)}{r^2} + \hat{c}_0$, $M = \frac{C(\lambda \log r + \theta)}{r^2} + \frac{\hat{c}_0}{2\lambda} \int^{\lambda \log r + \theta} \hat{C}(s)\, ds + \frac{\hat{c}_0^2}{8}(1 + \lambda^{-2}) r^2 + c_0$, $(s = 1)$.

$$\tau = \delta_1 e^{-\hat{c}_0/\lambda t} \qquad \xi_1 = -\frac{\hat{c}_0 \delta_1}{2} e^{-\hat{c}_0/\lambda t} \qquad \xi_0 = \eta_0 = 0$$

$$\phi_1 = -\frac{\hat{c}_0 \delta_1}{2\lambda} e^{-\hat{c}_0/\lambda t}\left(\frac{\hat{c}_0}{2}(\lambda + \lambda^{-1}) r^2 + \boldsymbol{r} \cdot \boldsymbol{A} - \lambda \boldsymbol{r} \wedge \boldsymbol{A} + \int^{\lambda \log r + \theta} \hat{C}(s)\, ds + \frac{2c_0 \lambda}{\hat{c}_0} - 1\right).$$

Case 7.1a: $B = \hat{C}(r)$, $M = C(r) + d\theta$, $(s = 1)$.

$$\tau = 0 \qquad \xi_1 = \gamma \qquad \xi_0 = \eta_0 = 0 \qquad \phi_1 = \gamma\left(\int^r r\hat{C}(r) - \boldsymbol{r} \wedge \boldsymbol{A} + dt\right)$$

with $\hat{C}(r) \neq \hat{C}_0 r^{-2} + \hat{c}_0$.

Case 7.1b: $B = \hat{C}(r) + 2\hat{d}\theta$, $M = C(r) + (\hat{d}\int^r r\hat{C}(r) + d)\theta + \frac{\hat{d}^2}{2}r^2\theta^2$, $(s = 1)$.

$$\tau = 0 \qquad \xi_1 = \gamma e^{\hat{d}t} \qquad \xi_0 = \eta_0 = 0 \qquad \phi_1 = \gamma e^{\hat{d}t}\left(\int^r r\hat{C}(r) - \boldsymbol{r} \wedge \boldsymbol{A} + \hat{d}r^2\theta + \frac{d}{\hat{d}}\right).$$

Case 8.1a: $B = \hat{C}(x)$, $M = by + c_0$, $(s = 1)$.

$$\tau = \xi_1 = \xi_0 = 0 \qquad \eta_0 = \beta_0 \qquad \phi_1 = \beta_0\left(A_2 - \int^x \hat{C}(x) - bt\right)$$

with $\hat{C}(x) \neq \hat{C}_0 x^{-2} + \hat{c}_0$.

Case 8.1b: $B = \hat{C}(x)$, $M = \frac{c^2}{2}y^2 + (c\int^x \hat{C}(x) + b)y + c_0$, $(s = 1)$.

$$\tau = \xi_1 = \xi_0 = 0 \qquad \eta_0 = \beta_1 e^{ct} \qquad \phi_1 = \beta_1 e^{ct}\left(A_2 - cy - \int^x \hat{C}(x) - \frac{b}{c}\right)$$

with $\hat{C}(x) \neq \hat{C}_0 x^{-2} + \hat{c}_0$.

4. Example

In this section we illustrate the construction of group-invariant solutions for a physically significant family of Fokker–Planck equations (2) with irrotational drift vectors. Let

$$A(x, y) = a_1 \log x + a_2 y$$

where $a_1 > 0$ and a_2 are constants. The Fokker–Planck equation (2) with drift vector

$$\boldsymbol{A} = \nabla A = \left(\frac{a_1}{x}, a_2\right) \tag{26}$$

describes the motion of a Brownian particle subject to the force $\boldsymbol{A}$, [2]. It follows from equation (21) that the corresponding function M is

$$2M = \frac{a_1(a_1 - 1)}{x^2} + a_2^2.$$

Therefore, M belongs to the case 1.1a of our classification, with

$$C_0 = \frac{a_1(a_1 - 1)}{2} \qquad b = 0 \qquad c_0 = \frac{a_2^2}{2}.$$

Taking one of the parameters δ_1, δ_2, β_1 and β_0 equal to one and the rest equal to zero in the formulae for τ, η_0 and φ of case 1.1a and using equation (20), we obtain the nontrivial symmetry generators

$$X_4 = \frac{x}{2}\partial_x + \frac{y}{2}\partial_y + t\partial_t + \frac{1}{2}(a_1 + a_2 y - a_2^2 t)u\partial_u$$

$$X_5 = tx\partial_x + ty\partial_y + t^2\partial_t + \left(-\frac{r^2}{2} + t\left(a_1 - 1 + a_2 y - \frac{a_2^2}{2}t\right)\right)u\partial_u$$

$$X_6 = t\partial_y + (a_2 t - y)u\partial_u$$

$$X_7 = \partial_y + a_2 u\partial_u.$$

Our purpose is to obtain explicit solutions by symmetry reduction of the Fokker–Planck equation (2) with drift vector (26). As explained in [4, 5], in order to reduce the PDE (2) to an ODE we need to find a two-dimensional subgroup of the whole symmetry group with two-dimensional orbits in the space of independent variables $(\boldsymbol{r}, t)$. An interesting such subgroup is generated by the subalgebras $\mathfrak{g}_\nu = \langle(\nu - a_2)X_2 + 2X_4, X_6\rangle$, where $X_2 = u\partial_u$ and ν is a real parameter. A complete set of local invariants under the action of this subgroup are

$$z = \frac{x}{\sqrt{t}} \qquad v = t^{-\nu} \exp\left(\frac{(y - a_2 t)^2}{2t}\right) u.$$

According to the general theory, a group-invariant solution takes the form

$$u(x, y, t) = t^\nu \exp\left(-\frac{(y - a_2 t)^2}{2t}\right) v(z).$$

Inserting this expression into the Fokker–Planck equation (2) with drift vector (26) we obtain the reduced ODE for $v(z)$, namely

$$v'' + \left(z - \frac{2a_1}{z}\right) v' + \left(\frac{2a_1}{z^2} - (2\nu + 1)\right) v = 0 \tag{27}$$

where the prime denotes the derivative with respect to z. The general solution of this equation may be expressed in terms of confluent hypergeometric functions. Indeed, the function $w(\zeta)$ defined by

$$v(z) = z^{2a_1} e^{-z^2/2} w(z^2/2)$$

satisfies Kummer's equation

$$\zeta w_{\zeta\zeta} + (a_1 + \tfrac{1}{2} - \zeta) w_\zeta - (\nu + 1) w = 0.$$

The general $\mathfrak{g}_\nu$-invariant solution of the Fokker–Planck equation (2) with drift vector (26) is therefore

$$u = t^{\nu + a_1} x^{2a_1} e^{-\frac{x^2 + (y - a_2 t)^2}{2t}} \left(c_1 M\left(\nu + 1, a_1 + \frac{1}{2}, \frac{x^2}{2t}\right) + c_2 U\left(\nu + 1, a_1 + \frac{1}{2}, \frac{x^2}{2t}\right)\right)$$

where $M(a, b, \zeta)$ and $U(a, b, \zeta)$ are Kummer's functions [15]. If $\nu + 1 = -n$, with n a non-negative integer, and $a_1 - \frac{1}{2}$ is not an integer, the general solution is

$$u = t^{a_1 - n - 1} x^{2a_1} e^{-\frac{x^2 + (y - a_2 t)^2}{2t}} \left(c_1 L_n^{(a_1 - \frac{1}{2})}\left(\frac{x^2}{2t}\right) + c_2 (\frac{x^2}{t})^{\frac{1}{2} - a_1} M\left(\frac{1}{2} - n - a_1, \frac{3}{2} - a_1, \frac{x^2}{2t}\right)\right)$$

where $L_n^{(a_1 - \frac{1}{2})}$ are generalized Laguerre polynomials. If we look for square-integrable solutions with no space singularities we must take $c_2 = 0$ in the above expression.

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